Kuwait University
Department of Mathematics

Math 101 — Calculus I	Spring Semester 2010	In-term Test 2

Sunday 9 May 2010

Duration: 90 minutes Total marks: 25

Calculators, pagers, and telephones are NOT allowed.
Answer all the following questions. Read each question carefully.

1. Find an equation of the tangent line to the curve

$$x^2y^2 - 2x = 4 - 4y$$

at the point $(2, -2)$. [4 marks]

2. Find the absolute extreme values of

$$f(x) = x\sqrt{4 - x^2} \quad \text{for } -1 \leq x \leq 2.$$ [4 marks]

3. Let

$$f(x) = \sin^2(x^2 - 3x + \pi/4).$$

   (a) Find $f'(x)$. [2 marks]

   (b) Use a linear approximation to estimate $f(0.02)$. [2 marks]

4. A meeting ends at 6.00 p.m. Amal leaves immediately, and drives North at a speed of 80 km/hour. Batool leaves a 1/2 hour later, and drives East at a speed of 120 km/hour. How fast is the distance between them increasing at 7.00 p.m.? [4 marks]

5. Use the Mean Value Theorem to show that

$$\tan(3/2) - \tan(1/2) > 1.$$ [2 marks]

6. Let

$$f(x) = \frac{x^2 - x - 2}{(x-1)^2}.$$

Find the intervals on which f is increasing or decreasing. [2 marks]

7. A rational function f has the following properties.

- Its domain is $(-\infty, -1) \cup (-1, \infty)$.
- $f(-2) = 0$, $f(0) = 2$, $f(1) = 1$, $f(2) = 3$.
- $\lim_{x\to -1^-} f(x) = \infty$ and $\lim_{x\to -1^+} f(x) = \infty$.
- $\lim_{x\to -\infty} f(x) = -\infty$ and $\lim_{x\to \infty} f(x) = \infty$.
- f is increasing on $(-\infty, -1)$ and $(1, \infty)$.
  f is decreasing on $(-1, 1)$.
- The graph of f is concave upward on $(-\infty, -1)$ and $(-1, 2)$.
  The graph of f is concave downward on $(2, \infty)$.

   (a) Find the local maxima and minima of f, if any. [1 mark]

   (b) Find the points of inflection on the graph of f, if any. [1 mark]

   (c) Sketch the graph of f. [3 marks]

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ANSWERS

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1. By implicit differentiation,

$$2xy^2 + 2x^2y\frac{dy}{dx} - 2 = -4\frac{dy}{dx}.$$

This gives

$$\frac{dy}{dx} = \frac{1 - xy^2}{x^2y + 2}.$$

Hence, at the point $(x, y) = (2, -2)$, the slope of the tangent line is

$$\frac{dy}{dx} = \frac{1 - 2(-2)^2}{2^2(-2) + 2} = \cdots = \frac{7}{6}.$$

An equation of the tangent line in point-slope form is therefore

$$y - (-2) = \tfrac{7}{6}\,(x - 2).$$

This simplifies to

$$y = \tfrac{7}{6}\,x - \tfrac{13}{3} \qquad \text{or} \qquad 7x - 6y - 26 = 0.$$

2. By the product rule and the chain rule,

$$f'(x) = \sqrt{4 - x^2} + x\frac{-x}{\sqrt{4 - x^2}} = \cdots = 2\frac{2 - x^2}{\sqrt{4 - x^2}} = 2\frac{\left(\sqrt{2} + x\right)\left(\sqrt{2} - x\right)}{\sqrt{(2 + x)(2 - x)}}.$$

Therefore, $\sqrt{2}$ is the only critical number of f in the interval $(-1, 2)$. So, by the Extreme Value Theorem, the absolute extrema of f are at -1, $\sqrt{2}$ or 2. Substitution shows that

$$f(-1) = -\sqrt{3}, \quad f(\sqrt{2}) = 2, \qquad \text{and} \qquad f(2) = 0.$$

Answer: The absolute maximum value of f is 2, and the absolute minimum value is $-\sqrt{3}$.

3. (a) By a double-angle formula,

$$f(x) = \tfrac{1}{2} - \tfrac{1}{2}\cos\big(2x^2 - 6x + \pi/2\big).$$

Hence, by the chain rule,

$$\begin{aligned} f'(x) &= \tfrac{1}{2}\sin\big(2x^2 - 6x + \pi/2\big)\frac{d}{dx}(2x^2 - 6x) \\ &= \tfrac{1}{2}\sin\big(2x^2 - 6x + \pi/2\big)\,(4x - 6) \\ &= (2x - 3)\sin\big(2x^2 - 6x + \pi/2\big). \end{aligned}$$

(b) The linear approximation of f at 0 is

$$f(x) \approx f(0) + f'(0)\,(x - 0)\,.$$

Since,

$$f(0) = \tfrac{1}{2} - \tfrac{1}{2}\cos(\pi/2) = 0.5 \quad \text{and} \quad f'(0) = (-3)\sin(\pi/2) = -3,$$

this gives

$$f(0.02) \approx 0.5 - 3\,(0.02) = 0.44.$$

4. Let x denote the distance travelled by Amal after the meeting (in km).
Let y denote the distance travelled by Batool after the meeting (in km).
Let z denote the distance between Amal and Batool (in km).
Let t denote time (in hours).
At 7.00 p.m.,

$$x = 80\,(1) = 80,$$

$$y = 120\,(\tfrac{1}{2}) = 60,$$

$$z = \sqrt{x^2 + y^2} = \sqrt{(80)^2 + (60)^2} = 20\sqrt{4^2 + 3^2} = \cdots = 100,$$

and

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2 + y^2}} = \frac{80(80) + 60(120)}{100} = 8(8) + 6(12) = \cdots = 136.$$

Answer: At 7.00 p.m., the distance between Amal and Batool is increasing at a rate of 136 km/hour.

5. Since the tangent function is continuous on $[1/2, 3/2]$ and differentiable on $(1/2, 3/2)$, the Mean Value Theorem implies that

$$\tan(3/2) - \tan(1/2) = \left(\frac{d}{dx}\tan x\right)\bigg|_{x=c}\left(\frac{3}{2} - \frac{1}{2}\right) = \sec^2 c \quad \text{for some } c \in (1/2, 3/2).$$

However,

$$\sec c > 1 \quad \text{for every } c \in (1/2, 3/2).$$

6. The domain of f is $(-\infty, 1) \cup (1, \infty)$. By the quotient rule and the chain rule,

$$\begin{aligned} f'(x) &= \frac{(2x-1)(x-1)^2 - 2(x^2-x-2)(x-1)}{(x-1)^4} \\ &= \frac{(2x-1)(x-1) - 2(x^2-x-2)}{(x-1)^3} = \cdots = \frac{5-x}{(x-1)^3}. \end{aligned}$$

Hence, 5 is the only critical number of f.

Interval	$(-\infty, 1)$	$(1, 5)$	$(5, \infty)$
$5-x$	$+$	$+$	$-$
$x-1$	$-$	$+$	$+$
$f'(x)$	$-$	$+$	$-$
f	decreasing	increasing	decreasing

Answer: f is increasing on $(1, 5]$, and, f is decreasing on $(-\infty, 1)$ and $[5, \infty)$.

7. (a) f has a local minimum at 1 and no other local extrema.

   (b) The graph of f has a point of inflection at $(2, 3)$ and no others.

   (c)